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Model order reduction of unsteady flow past oscillating airfoil cascades

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Abstract

This study focuses on developing reduced-order models for unsteady aerodynamic flows past a cascade of two-dimensional airfoils. A reduction method known as System Equivalent Reduction Expansion Process (SEREP) is used. The computational efficiency of the SEREP reduced-order model is compared with a reduced-order model formed using the Proper Orthogonal Decomposition (POD) technique. The present study shows that the SEREP is computationally more efficient than POD.

1. Introduction

High fidelity aerodynamic models usually involve large number of system degrees of freedom. Large order unsteady aerodynamic models are not suited for stability analysis or control design of aeroservoelastic systems. Therefore, it is desirable to reduce the model order with fewer states but which can still represent the system aerodynamics reasonably accurately.

A commonly used reduction technique is based on projecting the reference aerodynamic model onto a reduced-order subspace or basis. These basis vectors are chosen carefully so that the system dynamics is accurately captured by a minimum number of states. One way to construct the basis vectors is to use the system eigenvectors obtained from linearized system matrices. Eigenmodes are well known for their use of representing system responses over a given frequency range, and have been widely used in structural dynamic problems (Maia and Silva, 1997). Modal analysis techniques have been used in fluid flow problems too, though it is relatively recent (Dowell et al., 1997). Hall et al. (1995) have computed the natural frequencies and mode shapes of unsteady compressible flows through airfoil cascades and used the eigenmodes to form a reduced-order model. Hall (1994) used the natural modes of incompressible unsteady flows past two dimensional airfoil and airfoil cascades, and three-dimensional wings, to construct reduced-order models. An alternate approach to the eigenmode based reduction is the proper orthogonal decomposition (POD) or Karhunen–Loève method for extracting modal information based on simulations of the system (Berkooz et al., 1993) at different time or frequency instances. Newman (1996a, b) has discussed the theoretical background and application of this reduction technique. Romanowski (1996) used POD to reduce the model order of a 2-D compressible Euler solution of flow past an oscillating airfoil. Kim (1998) has applied this technique in the frequency domain for 3-D vortex lattice model of unsteady incompressible potential flow past a wing. More recently Epureanu et al. (2000, 2001) have applied this technique to viscous-inviscid flow simulations past airfoil cascades. Willcox (2000) has used this technique together with eigenmode based reduction as well as Arnoldi vector based reduction of unsteady aerodynamic model of inviscid compressible flow past airfoil cascades.

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Nomenclature

c	chord length of the airfoil
k	reduced frequency, $\omega c/V_\infty$
V_∞	free-stream velocity
Γ	vorticity strength of body and wake vortices
ω	frequency of oscillation, Im(λ)

Proper orthogonal decomposition has its advantages in that it can be used for model order reduction of linear as well as nonlinear systems. However, eigenmodes based reduction schemes have certain advantages in that choice of modes and degrees of freedom to be retained is easily decided by the frequency range of interest and the spatial participation in the selected modes in the frequency range. Also, they are relatively inexpensive in terms of computer effort in comparison to POD reduction technique (Parte, 2002).

In the present study we implement an eigenmode based reduction model known as System Equivalent Reduction Expansion Process (SEREP). SEREP has been used in structural dynamics for linear as well as nonlinear systems with localized nonlinearities (O'Callahan et al., 1989; Avitabile et al., 1989). The eigenmodes are chosen depending on the desired frequency spectrum over which the airfoil cascade oscillate. The degrees of freedom are retained depending on their significant participation in the chosen modes. This reduction method is illustrated on numerical lumped vortex panel implementation of a two-dimensional incompressible, potential flow past a cascade of NACA 0012 airfoils. We have also compared this technique with POD based reduction in terms of computational cost.

2. Equations of motion

The two-dimensional vortex lattice model for airfoil cascades is discussed in this section (Katz and Plotkin, 2001). The airfoils and their wakes are represented by a number of vortices. All the wake vortices are equidistant from each other in the stream-wise direction. The airfoil is divided into a number of panel elements and discrete point vortices are placed at the control points, located at one-fourth distance of each panel. The collocation points are assumed to be at the three-fourth distance of each panel. The downwash velocity induced by the body and wake vortices of the airfoils in the cascade are calculated by Whitehead's method (Whitehead, 1960). It assumes that the downwash at the collocation points of an airfoil is shifted in phase by a constant interblade phase angle σ from the downwash of the airfoil immediately next to it in the cascade. Thus, the vortex strength on the body and the wake for all the airfoils are identical but shifted by a phase angle $m\sigma$, where m is the blade number with respect to a reference blade. The downwash at a collocation point on the reference blade is a summation of all the vortices on the reference blade body and wake and also their phase shifted counterparts on other blades. The downwash is given by

$$(w_i)_k = \sum_{j=1}^{N} K_{ij}(\gamma_j)_k, \quad i = 1, \ldots, M, \tag{1}$$

where K_{ij} is called the kernel function of the downwash of cascade, $(\gamma_j)_k$ is the strength of the jth body vortex at kth instant of time, and $(w_i)_k$ is the downwash at the ith collocation point at kth instant of time. M is the number of the body vortex points and N is the total number of the body and wake vortex points. The kernel function K_{ij} is given by (Whitehead, 1960)

$$K_{ij} = W(x_i - \xi_i) - W(-\infty), \tag{2}$$

$$W(z) = \frac{1}{4s}\left\{\frac{\exp[-(\pi-\sigma)\exp(\mathrm{i}\theta)z/s + \mathrm{i}\theta]}{\sinh[\pi\exp(\mathrm{i}\theta)z/s]} + \frac{\exp[(\pi-\sigma)\exp(-\mathrm{i}\theta)z/s - \mathrm{i}\theta]}{\sinh[\pi\exp(-\mathrm{i}\theta)z/s]}\right\}, \tag{3}$$

where θ is the stagger angle, s is the interblade gap, and σ $(0\leqslant\sigma\leqslant 2\pi)$ the interblade phase angle.

The wake vortices are carried away by the free stream velocity and the distance between one wake vortex to the other in the stream-wise direction is given by $\Delta x = V_\infty \Delta t$, where Δt is the time step and V_∞ is the free stream velocity. The first wake vortex or the shed vortex has the strength of the time rate of change of the circulation about the airfoil. That is,

$$(\Gamma_{M+1})_{k+1} = -\sum_{j=1}^{M} [(\gamma_j)_{k+1} - (\gamma_j)_k]. \tag{4}$$

The shed vorticity is convected into the wake by the speed of the free-stream velocity. The vorticity convection in the wake is expressed by

$$(\Gamma_i)_{k+1} = (\Gamma_{i-1})_k, \quad i = M+2, \ldots, N-1. \tag{5}$$

In order to form the governing equations, the downwash induced at the collocation points due to wake and body vortices are calculated. The boundary condition on the airfoil surface is the no normal flow condition at the body collocation points. Total normal flow due to vorticity downwash and due to free-stream and airfoil motion are added and equated to zero. This gives M boundary conditions. They are combined with the wake vortex convection equations to form the governing equations. The governing equations are given in the form of a time difference equation,

$$\boldsymbol{A\Gamma}_{k+1} + \boldsymbol{B\Gamma}_k = \boldsymbol{v}_{k+1}, \tag{6}$$

where A and B are $N \times N$ matrices, Γ is the unknown vorticity vector containing the body and wake vortices, $\boldsymbol{v}_{k+1}$ is the vector of normal velocities due to free-stream and airfoil motion at the $(k+1)$th time instant. The generalized eigenvalue problem is solved by setting the right-hand side equal to zero. The associated eigenvalue problem is then

$$(zA + B)X = 0, \tag{7}$$

where $z = \exp(\lambda \Delta t)$ are the discrete time eigenvalues, and X is the eigenvector.

3. Reduced-order model using SEREP

We briefly review SEREP. A few of the eigenmodes and degrees of freedom from the original model are retained. Only those eigenmodes and degrees of freedom whose contribution in the system dynamics is significant, need to be retained. The natural frequencies and mode shapes of the reduced model will be exactly the same as that of the retained modes from the original model. Further, the eigenvalues of the SEREP reduced-order model is independent of the choice of retained degrees of freedom. However, the choice of degrees of freedom is important from the point of view of forced response analysis using the reduced-order model in that the spatio-temporal characteristics of the forcing may selectively excite different modes at different degrees of freedom.

Consider a general dynamical system in the state space form:

$$\dot{x} = Fx, \tag{8}$$

where x is the state vector with n unknowns. The modal expansion of x is given by

$$x_n = U_{nn} p_n, \tag{9}$$

where U_{nn} is the modal matrix of order $n \times n$ and p_n is the vector of modal coordinates. Retaining the important modes for system dynamics, say m modes, we get

$$x_n = U_{nm} p_m. \tag{10}$$

Now, to form a reduced-order model with a number of system unknowns, the number of degrees of freedom to be chosen is a and these are called active degrees of freedom. Let the number of deleted degrees of freedom be denoted by d. Then,

$$x_n = [x_a^{\mathrm{T}} \;\; x_d^{\mathrm{T}}]^{\mathrm{T}} = [U_{am}^{\mathrm{T}} \;\; U_{dm}^{\mathrm{T}}]^{\mathrm{T}} p_m, \tag{11}$$

where $(\cdot)^{\mathrm{T}}$ refers to transpose of a matrix.

Therefore,

$$p_m = U_{am}^g x_a, \tag{12}$$

where the superscript g refers to a generalized inverse or pseudo-inverse as matrix U_{am} may not be square in general. There are two possible cases. For the case when $a > m$, the generalized inverse is of rank m and is given by $U_{am}^g = (U_a^{\mathrm{T}} U_a)^{-1} U_a^{\mathrm{T}}$. However, for the case when $a < m$, that is the number of equations are less than the number of unknowns, the generalized inverse is of rank a, and is given by $U_{am}^g = U_a^{\mathrm{T}} (U_a U_a^{\mathrm{T}})^{-1}$. This produces only an average solution for the m modal displacements and can lead to incorrect solutions (O'Callahan et al., 1989). However, this situation is very unlikely to occur in the use of SEREP since usually the number of retained degrees of freedom a is usually much larger than the number of retained modes m. Now returning to the development of the SEREP methodology, we substitute Eq. (12) into Eq. (10) and obtain

$$x_n = U_{nm} U_{am}^g x_a = T x_a. \tag{13}$$

T is the transformation matrix relating the original state vector x_n to the reduced-order state vector x_a. Substituting this transformation in Eq. (8), one gets

$$\dot{x}_a = T^{-1}FTx_a. \tag{14}$$

The eigenvalues of this reduced-order system of unknowns of order a is exactly the same to those m retained modes of the original n order system. Further, unlike other reduction techniques, SEREP is a reversible process. That is, one can compute the original system matrices from the reduced order ones using the transformation matrix T defined by Eq. (13).

4. Proper orthogonal decomposition

Proper orthogonal decomposition (POD) is a popular approach for determining reduced-order models of dynamical systems. Typically, a time simulation of the system is performed and instantaneous solutions or 'snapshots' are obtained at selected times. A two-point time correlation matrix is constructed from these data. The eigenvectors of this correlation matrix then form the orthogonal set of basis vectors which will represent the solution in an optimal way (Berkooz et al., 1993; Dowell and Hall, 2001).

Let the instantaneous flow field at the ith time instant be defined as $v^{(i)}(x) = v(x, t_i)$. Then the two-point correlation function C_{ij} be defined as

$$C_{ij} = \int_R v^{(i)}(x)v^{(j)}(x)\,dx, \quad i,j = \text{time or frequency points}, \tag{15}$$

where R is the spatial domain. The eigenvectors $A^{(n)}$ and the corresponding eigenvalues λ_n of C satisfy

$$CA^{(n)} = \lambda_n A^{(n)}. \tag{16}$$

The empirically determined eigenfunctions are then computed as linear combinations of the data snapshots, using

$$\phi_n(x) = \sum_{k=1}^{M} A_k^{(n)} v^{(k)}(x), \quad n = 1, 2, \ldots, M. \tag{17}$$

Then, the system response can be represented in terms of these eigenfunctions as

$$v(x, t) = \sum_{n=1}^{M} a_n(t)\phi_n(x), \tag{18}$$

where $a_n(t)$ are time-dependent coefficients. The Karhunen–Loève expansion theorem (Berkooz et al., 1993, p. 545) states that the decomposition Eq. (18) is optimal for any square-integrable signal. Besides, the time functions $a_n(t)$ are uncorrelated too. The representation of the system response in terms of the K–L modes, Eq. (18), is essentially the projection of the system response to a reduced-order solution space. Applying this transformation on the full order system equations, we get a set of equations of motion for the reduced-order system. As shown by Kim (1998), POD can be implemented in the time or the frequency domain.

5. Results and discussion

The airfoils in the cascade are NACA 0012 sections. As described in Section 2, a discrete vortex panel was used to model flow past the airfoil cascades (Katz and Plotkin, 2001). The airfoils are assumed to be undergoing pitching motion. The cascade parameters are: stagger angle $\theta = 45^\circ$; inter-blade spacing to airfoil chord ratio $s/c = 1$; inter-blade phase angle $\sigma = 72^\circ$. The flow past the airfoil is modeled using 20 vortex elements. The wake is discretized with 200 free vortex elements modeled upto 200 time steps. These wake points stretched up to a length of 10 chords behind the airfoil.

The generalized eigenvalue problem, Eq. (7), is solved for the eigenvalues and eigenvectors. The matrices $[A]$ and $[B]$ in Eq. (7) are in general large, sparse and unsymmetric. Matrix $[A]$ is in general complex. The eigenvalues in the z- and λ-plane are shown in Fig. 1. The imaginary part of λ corresponds to the reduced frequency k. We first consider the SEREP technique for model order reduction by retaining different set of modes and degrees of freedom. We consider the unsteady flow past the pitching airfoil cascade at $k = 1.0$. The first 40 eigenmodes were retained. As shown in Fig. 1, this frequency of oscillation lies well within these retained modes. The results of these simulations are shown in Fig. 2. The number of retained degrees of freedom is kept fixed at 40; U_{am} is therefore square so that it can be

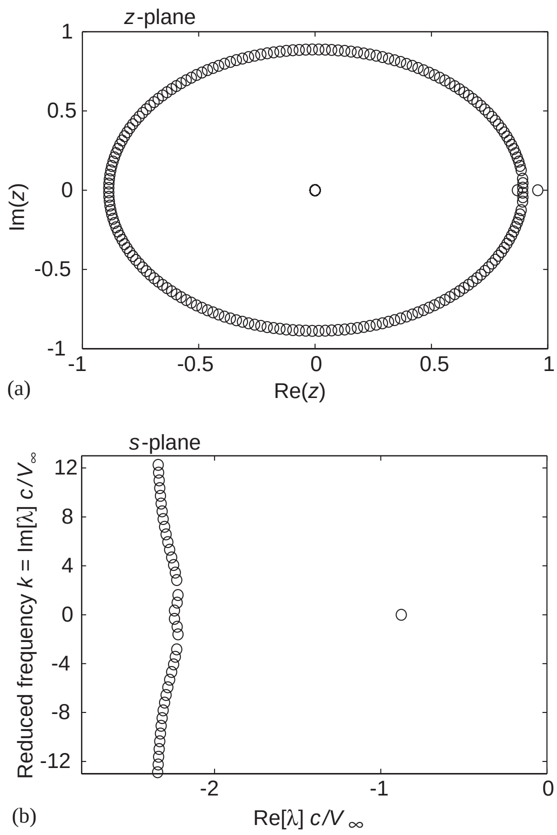

Fig. 1. Eigenvalues in z- and λ-plane.

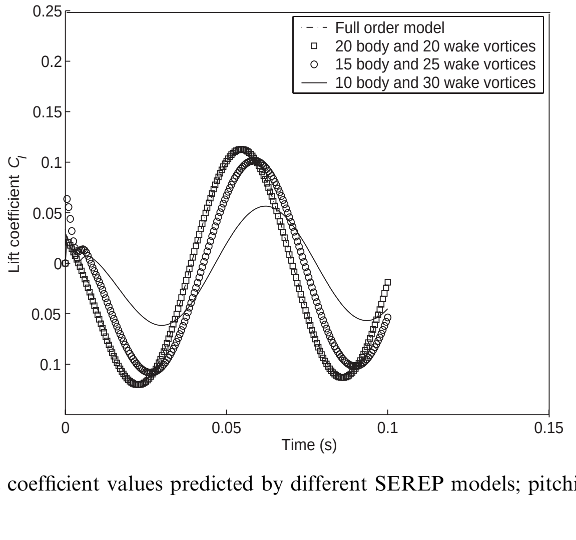

Fig. 2. Comparison of the lift coefficient values predicted by different SEREP models; pitching airfoil cascade at $k = 1.0$.

inverted in the usual way. However, we varied the location of the degrees of freedom keeping the total number fixed at 40. In Fig. 2 it is seen that retaining all the airfoil surface vortex points is necessary for the reduced-order model to follow the system dynamics closely. Since the flow over the airfoil was modeled with 20 discrete vortices, all 20 airfoil vortices were retained. A minimum of 20 wake vortices were needed to complete an accurate reduced-order model. We experimented with lower number of modes than 40, but the resulting reduced-order models were not able to follow the system dynamics accurately.

Although the aforedescribed reduced-order model reproduced the system response accurately at $k = 1.0$, an obvious question to ask is whether this reduced-order model will accurately capture the system response for other frequencies of excitation of the pitching airfoils in the cascade. We simulated this reduced-order model airfoil oscillation frequencies beyond $k = 1$, and it was able to follow the dynamics of the full order model till $k = 1.6$. For frequencies outside this range, obviously a different set of modes need to be chosen.

We were also interested in evaluating the reduced-order model response to a more generalized motion of the pitching airfoils. So, we consider step-input type excitation. The reduced-order model discussed above, with 40 modes and degrees of freedom, did not accurately capture the response. After few simulations, a reduced-order model with the first 50 eigenmodes was found adequate. As before, 50 degrees of freedom were retained, with 20 of them over the airfoil surface and 30 in the wake. This reduced-order model accurately followed the system response to the step input. This is shown in Fig. 3. Furthermore, this reduced-order model reproduces the system response well when the airfoil cascade was pitching harmonically in a reduced frequency range $k = 0.1$–3.0, as shown in Fig. 4. This figure shows the maximum lift coefficient at different reduced frequency values in the range of $k = 0.1$–4.0 and a good match is observed between the 50 mode reduced model and full order model upto $k = 3.0$.

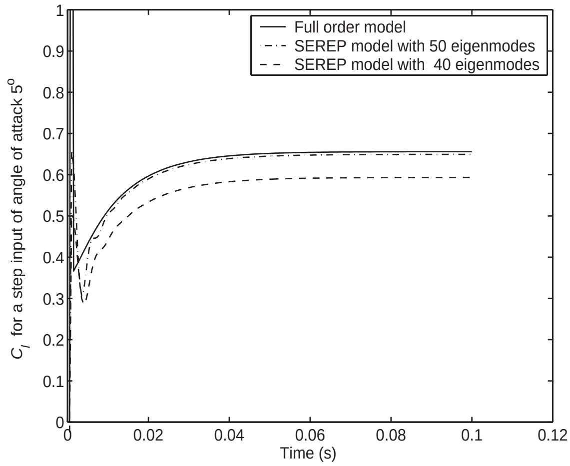

Fig. 3. Lift coefficient time-history predicted by different SEREP models due to step disturbance of airfoil cascade.

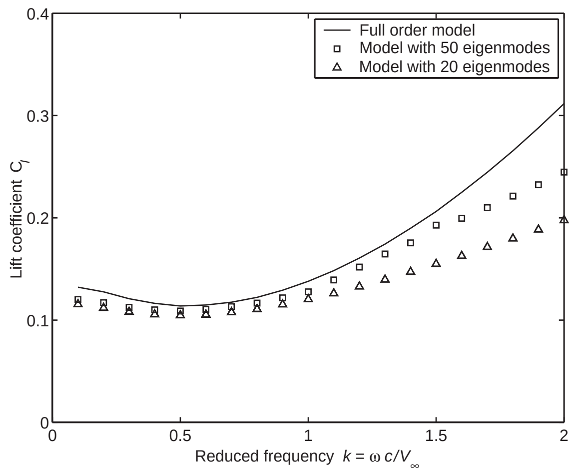

Fig. 4. Lift coefficient amplitude as a function of reduced frequency k predicted by different SEREP models.

We now discuss the results obtained using a reduced-order model based on the proper orthogonal decomposition (POD) technique. A frequency domain implementation of the POD method was used. The original system was simulated in the frequency domain at different frequency values. To do so, a z-transform of the governing discrete time equation, Eq. (6), was performed, and the solution was computed in the frequency domain at different frequencies from the chosen window. In the first instance, the simulation frequencies for the model was in the frequency interval $k = 0.1$–6.0 in steps of $\Delta k = 0.1$. This frequency window covered the oscillation frequency ($k = 1.0$) of the full-order model of the airfoil cascade. Solving the eigenvalue problem associated with the two-point correlation matrix C in Eq. (16), we retained the different sets of K–L modes to form the reduced-order model. After few simulations, the first 60 K–L modes was seen to be sufficient to accurately track the system dynamics. Fig. 5 shows the lift coefficient time-history computed with 60 K–L modes. Fig. 6 compares the result obtained using two different POD models having different simulation frequency windows but with the same number of retained K–L modes. It is evident that the results are significantly different for different choice of simulation frequencies. Note that in both the models the simulation frequencies cover the actual model oscillation frequency.

We also investigated the effectives of the POD reduced-order model to predict the system response over a frequency range other than $k = 1.0$. Snapshots were generated in the frequency domain at reduced frequencies $k = 0.02$–14 in steps of $\Delta k = 0.02$. Solving the eigenvalue problem for the correlation matrix formed with the snapshots, this gave a total of 1400 POD modes. The first 20, 40, and 60 POD modes were considered to form reduced-order models. From

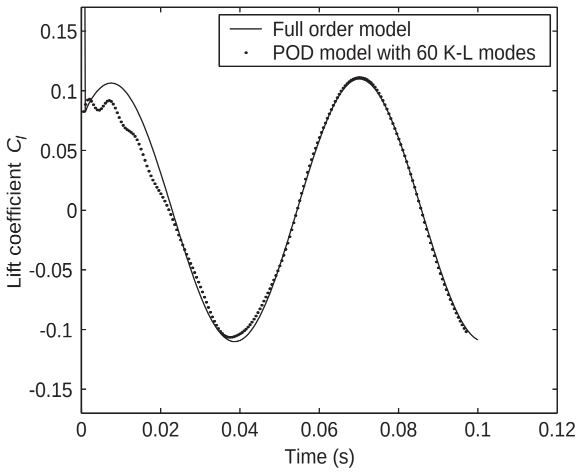

Fig. 5. Lift coefficient predicted by POD model with 60 K–L modes.

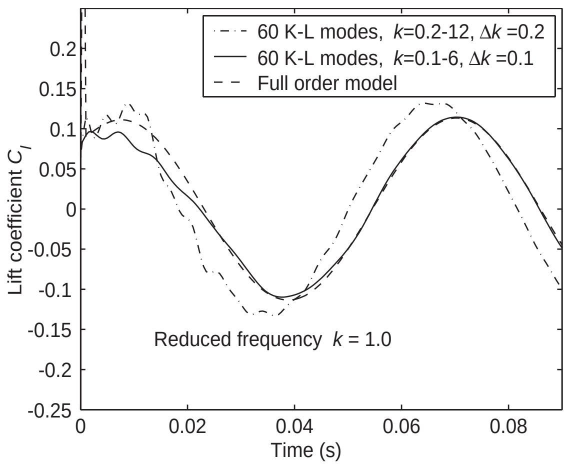

Fig. 6. Lift coefficient predicted by different POD models using different simulation frequency interval and frequency step-size, but with same number of retained K–L modes.

Fig. 7 it is seen that the POD model with first 40 POD modes follows the system dynamics closely in the frequency range $k = 0.1$–3.0. The response of these POD reduced-order models to step input is shown in Fig. 8. Here, the POD model with 60 K–L modes seems to follow the system dynamics closely as compared to that formed with 40 K–L modes. In general though, POD models formed with snapshots of the time domain response of the system subjected to an impulse type input would be most suitable to form reduced-order POD models (Parte, 2002).

An issue of considerable significance while comparing reduced-order modeling approaches is the number of floating point operations needed to predict system response. Towards that, we have compared the number of floating point operations needed to simulate the dynamics of the full-order model, SEREP reduced-order model, and POD reduced-order model, in the time domain, for a fixed time interval with fixed time-step, and at a specified reduced frequency. The simulations were done using MATLAB. It was observed that the full order model took 1.29×10^{10} flops, whereas, SEREP and POD based reduced-order models takes 3.6×10^{8} and 3.45×10^{9} flops, respectively, to simulate through the same time window for an oscillation frequency $k = 1.0$. This shows that SEREP is more efficient by an order of magnitude. It is important to note that these values for the floating point operations pertain to the time-domain response calculations only. They do not include the floating point operations to form the reduced-order models. That is, in all the cases mentioned above, the flops count for SEREP does not consider the calculations pertaining to eigenanalysis and to form the transformation matrices. Elsewhere (Parte, 2002), it is shown that the number of floating

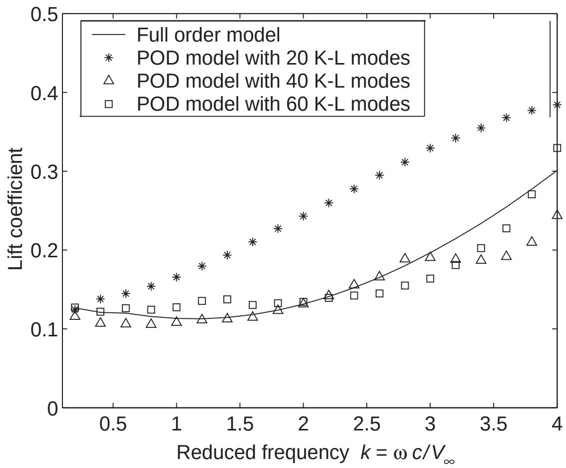

Fig. 7. Lift coefficient amplitude as a function of reduced frequency as predicted by POD models with different number of K–L modes.

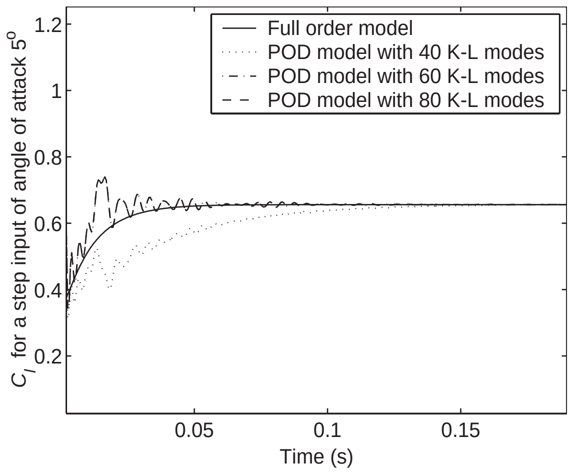

Fig. 8. Lift coefficient for step input predicted by POD reduced-order models.

point operations for these calculations are insignificant relative to the time-domain response calculations. For POD, the flops to generate the frequency domain snapshots and to form the K–L modes too have not been considered. However, in the case of POD reduced-order models, the number of floating point operations needed to generate these snap-shots are indeed very high, either in the time-domain or in the frequency domain, since the full-order system has to be simulated to generate these snapshots. The floating point operations needed in other parameter ranges have also been computed and SEREP is always seen to be more efficient than POD. The more number of operations for POD relative to SEREP could be attributed to the fact that SEREP reduces the number of degrees of freedom, whereas POD does not. In POD reduction, after simulating the system in terms of K–L modes, one always has to transform back to the original system coordinates.

6. Conclusions

In the present work we have applied System Equivalent Reduction Expansion Process (SEREP) to model order reduction for linear unsteady aerodynamic problems. This technique seems to be more efficient than Proper Orthogonal Decomposition (POD) in determining the response of the unsteady aerodynamic model of a flow past an oscillating airfoil cascade. Although we have chosen a lumped vortex model for the unsteady aerodynamics, which is usually of low order compared to finite element models, elsewhere (Parte, 2002) it is shown that SEREP is also efficient relative to POD in the case of finite element model reduction of incompressible potential flow. However, one should keep in mind that POD can be applied for model order reduction of nonlinear dynamic systems whereas SEREP is limited to linear systems.

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